九十五學年第一學期 PHYS2310 電磁學 期末考試題(共兩頁)

[Griffiths Ch. 5-6 & 1-4] 2007/01/09, 10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

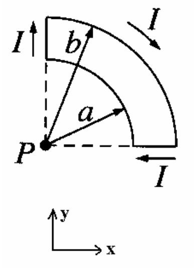
◇ Useful formulas: Spherical coordinate $\nabla\times(A_\phi\hat{\phi}) = \frac{1}{r\sin\theta}\frac{\partial(\sin\theta A_\phi)}{\partial\theta}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial(rA_\phi)}{\partial r}\hat{\theta}$

: Cylindrical coordinate $\nabla\times\mathbf{v} = [\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}]\hat{\mathbf{s}} + [\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s} -]\hat{\phi} + \frac{1}{s}[\frac{\partial(sv_\phi)}{\partial s} - \frac{\partial v_s}{\partial\phi}]\hat{\mathbf{z}}$

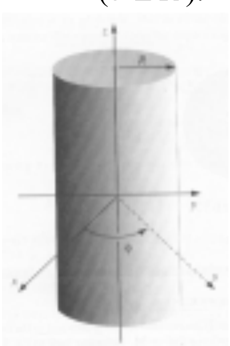
◇ Specify the magnitude and direction for a vector field

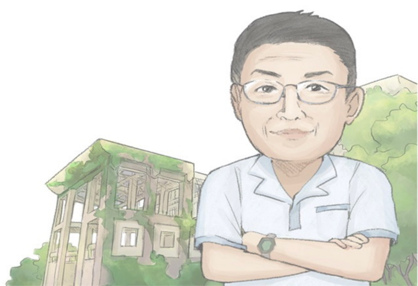
1. (8%,6%,6%) A *metal* sphere of radius R carries a uniform surface charge density σ, and is set spinning with angular velocity ω about the axis.
   (a) What is the magnetic dipole moment $\mathbf{m}$ of the sphere?
   (b) Find the approximate vector potential $\mathbf{A}$ at a point (r, θ) where $r \gg R$.
   (c) Find the magnetic field $\mathbf{B}$ at a point (r, θ) where $r \gg R$.

2. (10%, 10%) A steady current loop is placed in a uniform magnetic field as shown in the figure. The uniform magnetic field is $B_0\hat{\mathbf{z}}$.
   (a) Find the magnetic field $\mathbf{B}$ at point P generated by the loop.
   (b) Find the force $\mathbf{F}$ on the loop.

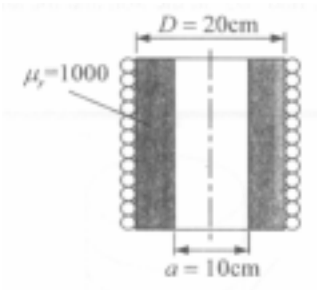


3. (10%, 10%) A long cylinder of radius R carries a magnetization $\mathbf{M}=M_0\hat{\mathbf{z}}$, where M_0 is a constant.
   (a) Find $\mathbf{J}_b$ within the material and $\mathbf{K}_b$ on the surface of the material.
   (b) Find the magnetic field $\mathbf{B}$ due to $\mathbf{M}$ for points inside ($r \le R$) and outside the cylinder ($r \ge R$).

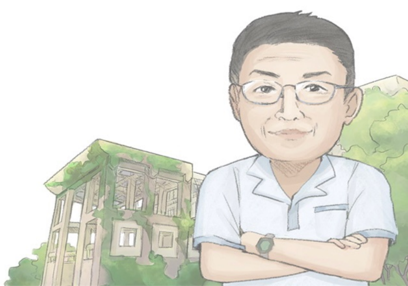




4. (7%, 7%, 6%) An infinite long magnetic material tube (μ_r=1000) of inner diameter a=10 cm and outer diameter D=20 cm is tightly wrapped with thin solenoid of 30 turns per unit cm, as shown in the figure. The current per turn I is 1 A. [Hint: $\mu_0 = 4\pi \times 10^{-7}$]
   (a) Find the auxiliary field **H** at the following three regions $0 \le r \le a/2$, $a/2 \le r \le D/2$, and $r \ge D/2$.
   (b) Find the magnetic field **B** at the following three regions $0 \le r \le a/2$, $a/2 \le r \le D/2$, and $r \ge D/2$.
   (c) Explain why the magnetic field **B** is discontinuous at the boundary $r = a/2$. [Hint: use the boundary condition for **B** field.]



5. (7%, 7%, 6%) Suppose the potential at the surface of a sphere is specified, $V(R_0, \theta) = V_0 \cos^2\theta$, where R_0 is the radius of the sphere and V_0 is a constant. There is no charge inside or outside the sphere.
   (a) Show that the potential outside the sphere.
   (b) Show that the electric field outside the sphere.
   (c) Show that the potential inside the sphere.
   [Hint: use Legendre polynomials, $P_0(x) = 1,\ P_1(x) = x,$ and $P_2(x) = (3x^2 - 1)/2$.]



1. Problems 5.56 + 5.58

(a) $a=\pi(R\sin\theta)^2=\pi R^2\sin^2\theta,\ dI=\dfrac{dQ}{2\pi/\omega}=\dfrac{2\pi(R\sin\theta)R\sigma\omega d\theta}{2\pi}=-R^2\omega\sigma d\cos\theta$

$$\mathbf{m}=-\hat{\mathbf{z}}\int_0^{\pi}\pi R^2\sin^2\theta R^2\omega\sigma d\cos\theta=-\hat{\mathbf{z}}\pi R^4\omega\int_1^{-1}(1-x^2)dx=\frac{4\pi}{3}R^4\sigma\omega\hat{\mathbf{z}}$$

(b) $\mathbf{A}_{\text{dip}}(\mathbf{r})=\dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\hat{\mathbf{r}}}{r^2}=\dfrac{\mu_0}{4\pi}\dfrac{4\pi}{3}R^4\sigma\omega\dfrac{\hat{\mathbf{z}}\times\hat{\mathbf{r}}}{r^2}=\dfrac{\mu_0}{3}R^4\sigma\omega\sin\theta\dfrac{\hat{\phi}}{r^2}$

(c)

$$\mathbf{B}_{\text{dip}}=\nabla\times\mathbf{A}_{\text{dip}}\qquad A_\phi=\frac{\mu_0}{3}R^4\sigma\omega\frac{\sin\theta}{r^2}$$

$$=\nabla\times(A_\phi\hat{\phi})=\frac{\mu_0}{3}R^4\sigma\omega\left[\frac{2\sin\theta\cos\theta}{r^3\sin\theta}\hat{\mathbf{r}}+\frac{\sin\theta}{r^3}\hat{\theta}\right]=\frac{R^4\sigma\omega\mu_0}{3r^3}\left[2\cos\theta\hat{\mathbf{r}}+\sin\theta\hat{\theta}\right]$$

2. Problems 5.9 + 5.10

(a) The straight segments produce no field at *P*.

The two quarter-circles gives: $\dfrac{\mu_0 I}{8}(\dfrac{1}{a}-\dfrac{1}{b})\hat{\mathbf{z}}$

(b)

$$\mathbf{F}_{\text{mag}}=-I\int(\mathbf{B}\times d\mathbf{l})=IB_0\left[(b-a)\hat{\mathbf{x}}+\int_{\pi/2}^{0}b(\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}})d\theta+(b-a)\hat{\mathbf{y}}+\int_0^{\pi/2}a(\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}})d\theta\right]$$

$$=IB_0\left[(b-a)\hat{\mathbf{x}}-\int_0^{\pi/2}b(\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}})d+(b-a)\hat{\mathbf{y}}+\int_0^{\pi/2}a(\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}})d\theta\theta\right]$$

$$=IB_0\left[(b-a)\hat{\mathbf{x}}+(-b+a)\hat{\mathbf{x}}+(b-a)\hat{\mathbf{y}}+(-b+a)\hat{\mathbf{y}}\right]$$

$$=0$$

3. Problems 6.8 + 6.9

(a) $\mathbf{J}_b=\nabla\times\mathbf{M}=0,\ \ \mathbf{K}_b=\mathbf{M}\times\hat{\mathbf{n}}=M_0\hat{\mathbf{z}}\times\hat{\mathbf{s}}=M_0\hat{\phi}$

(b) Use Ampere's law for the *solenoid-like* bar magnet

$$\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0\mathrm{I},\quad B\ell=\mu_0\ell K_b=\mu_0\ell M_0,\ \Rightarrow\ B=\mu_0 M_0,\quad \mathbf{B}=\mu_0 M_0\hat{\mathbf{z}}\ \text{ for }\ r\le R$$

$$\mathbf{B}=0\ \text{ for }\ r\ge R$$

4.

(a) $\oint\mathbf{H}\cdot d\mathbf{l}=\mathbf{I}_f$ (integral form) $H\ell=n\ell I,\ \Rightarrow\ H=nI=3000\cdot 1=3000$ in *z* direction

for $0\le r\le a/2$ and $a/2\le r\le D/2$. **H** field is uniform inside the solenoid.
$\mathbf{H}=0$ for $r\ge D/2$. **H** field is zero outside the solenoid.

(b)

$$B=\mu H=1000\cdot\mu_0 nI=\begin{cases}1\cdot 4\pi\times10^{-7}\cdot3000\cdot1=0.00377\text{ T}\ \text{ for } r\le 5\text{ cm}\\ 1000\cdot4\pi\times10^{-7}\cdot3000\cdot1=3.77\text{ T}\ \text{ for } 5\le r<10\text{ cm}\\ 0\ \text{ for } r>10\text{ cm}\end{cases}\ \text{in } z \text{ direction.}$$

(c) Due to bound surface current $\mathbf{K}_b$

$$\mathbf{M} = \frac{1}{\mu_0}\mathbf{B} - \mathbf{H} = (\mu_r - 1)\mathbf{H} \quad \text{and} \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}}$$

Boundary condition: $\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = \mu_0(\mathbf{K}\times\hat{\mathbf{n}})$

$$\mathbf{B}^{\parallel}_{\text{above}} - \mathbf{B}^{\parallel}_{\text{below}} = (u_r - 1)\mu_0\mathbf{H}$$

$$\mu_0(\mathbf{K}\times\hat{\mathbf{n}}) = \mu_0((\mathbf{M}\times\hat{\mathbf{n}})\times\hat{\mathbf{n}}) = (u_r - 1)\mu_0\mathbf{H}$$

5. First midterm, note $\sin^2\theta \;\;\Rightarrow\;\; \cos^2\theta$

(a) Boundary condition $\begin{cases} \text{(i) } V(R_0,\theta) = V_0\cos^2\theta \\ \text{(ii) } \lim\limits_{r\to\infty} V(r,\theta) = 0 \end{cases}$

General solution $V(r,\theta) = \sum\limits_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (ii) $\to A_\ell = 0$

B.C. (i) $\to B_0R_0^{-1} + B_1R_0^{-2}\cos\theta + B_2R_0^{-3}(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) = V_0\cos^2\theta$

$$\begin{cases} B_0R_0^{-1} - \tfrac{1}{2}B_2R_0^{-3} = 0 \\ B_1R_0^{-2} = 0 \\ \tfrac{3}{2}B_2R_0^{-3} = V_0 \end{cases} \Rightarrow \begin{cases} B_2 = \tfrac{2}{3}R_0^3V_0 \\ B_1 = 0 \\ B_0 = \tfrac{1}{2}B_2R_0^{-2} = \tfrac{1}{3}R_0V_0 \end{cases}$$

$$\therefore V(r,\theta) = \frac{R_0V_0}{3r} + \frac{2R_0^3V_0}{3r^3}P_2(\cos\theta)$$

(b) $V(r,\theta) = \dfrac{R_0V_0}{3r} + \dfrac{R_0^3V_0}{3r^3}(3\cos^2\theta - 1)$

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\boldsymbol{\theta}} = \left[\frac{R_0V_0}{3r^2} + \frac{R_0^3V_0}{r^4}(3\cos^2\theta - 1)\right]\hat{\mathbf{r}} - \frac{1}{r}\left[\frac{R_0^3V_0}{3r^3}(-6\cos\theta\sin\theta)\right]\hat{\boldsymbol{\theta}}$$

$$\mathbf{E} = \left[\frac{R_0V_0}{3r^2} + \frac{R_0^3V_0}{r^4}(3\cos^2\theta - 1)\right]\hat{\mathbf{r}} + \left[\frac{R_0^3V_0}{r^4}(\sin 2\theta)\right]\hat{\boldsymbol{\theta}}$$

(c)

Boundary condition $\begin{cases} \text{(i) } V(R_0,\theta) = V_0\cos^2\theta \\ \text{(ii) } \lim\limits_{r\to 0} V(0,\theta) \text{ is finite.} \end{cases}$

General solution $V(r,\theta) = \sum\limits_{\ell=0}^{\infty}(A_\ell r^\ell + B_\ell r^{-(\ell+1)})P_\ell(\cos\theta)$

B.C. (ii) $\to B_\ell = 0$

B.C. (i) $\to A_0 + A_1R_0^1\cos\theta + A_2R_0^2(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) = V_0\cos^2\theta$

$$\begin{cases} A_0 - \tfrac{1}{2}A_2R_0^2 = 0 \\ \tfrac{3}{2}A_2R_0^2 = V_0 \\ A_1R_0 = 0 \end{cases} \Rightarrow \begin{cases} A_2 = \tfrac{2}{3}R_0^{-2}V_0 \\ A_1 = 0 \\ A_0 = \tfrac{1}{3}V_0 \end{cases}$$

$$\therefore V(r,\theta) = \frac{V_0}{3} + \frac{2V_0}{3R_0^2}r^2P_2(\cos\theta)$$